Near-Horizon Conformal Structure of Black Holes

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Abstract

The near-horizon properties of a black hole are studied within an algebraic framework, using a scalar field as a simple probe to analyze the geometry. The operator H governing the near-horizon dynamics of the scalar field contains an inverse square interaction term. It is shown that the operators appearing in the corresponding algebraic description belong to the representation space of the Virasoro algebra. The operator H is studied using the representation theory of the Virasoro algebra. We observe that the wave functions exhibit scaling behaviour in a band-like region near the horizon of the black hole.

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1. Introduction

The relation between the physics of black holes and conformal field theory has been explored recently in a variety of contexts [1, 2, 3]. In particular, the near-horizon symmetry structure of general black holes in arbitrary dimensions (including the Schwarzschild case) has been studied [2, 3]. By imposing suitable boundary conditions at the horizon, it was shown that the relevant algebra of surface deformations contains a Virasoro algebra in the $(r-t)$-plane. This analysis is based on an extension of the Brown-Henneaux algebra of three-dimensional anti-de Sitter gravity [4]. In the latter case, purely classical considerations lead to the existence of an asymptotic symmetry algebra containing two copies of the Virasoro algebra.

In a separate line of development, it was found that the dynamics of particles or scalar fields near the horizon of a black hole is associated with a Hamiltonian containing an inverse square potential [5, 6, 7]. Since the scalar field can be viewed as a tool to probe the near-horizon geometry of the black hole, its dynamics should reveal any underlying symmetry of the system. Indeed, such a Hamiltonian was shown to have conformal symmetry quite some time ago [8], and this idea has been further explored recently [9, 5, 6].

In this paper, we provide a synthesis of the ideas appearing in the above approaches within an algebraic framework. Since our essential interest is in the near-horizon geometry of the black hole, it is useful to restrict attention to a very simple probe. Thus, we consider the time-independent modes of a scalar field in the black hole background. In particular, we study the case of a such a field in the background of a Schwarzschild black hole. The Klein-Gordon operator H governing the dynamics of the probe contains an inverse square potential term [7]. It is shown that H can be written in a factorized form, which leads to an algebraic description of the system in terms of the enveloping algebra of the Virasoro algebra. The inverse square interaction term plays a crucial role in obtaining this result. It may be noted that previous works in this direction did not treat the interaction term algebraically. Incorporating this term within the algebraic framework leads to a structure that contains approximately half of all the Virasoro generators. The requirement of a unitary representation of the resulting algebra allows us to include the remaining generators. We then describe the spectrum of H in terms of the wedge representations of the Virasoro algebra [10].

It should be noted that the inverse square term also plays an important role in determining the self-adjoint extensions of the Klein-Gordon operator [7]. In general, the corresponding wave functions violate scaling in the near-horizon region. However, we show that scaling behaviour is present in a small band-like region near the horizon, for certain choices of the self-adjoint extension. These self-adjoint extensions thus play a crucial role in providing a consistent picture of the whole analysis. The existence of this band region is reminiscent of the stretched horizon picture of black holes [11].

This paper is organized as follows. In section 2, we study the example of a scalar field probing the near-horizon properties of the Schwarzschild black hole. The operator governing the dynamics of the time-independent modes is written in a factorized form. It is shown that the resulting factors lead to an algebraic description in terms of the enveloping algebra of the Virasoro generators. Section 3 discusses the properties of this algebra in terms of the wedge representations of Ref. [10]. This leads to an algebraic description of the spectrum of the time-independent Klein-Gordon operator. The near-horizon scaling behaviour at the quantum

level is discussed in section 4. We conclude in section 5 with a brief discussion regarding the application of our results to more general black holes.

2. Algebraic Formulation of the Near-Horizon Dynamics

In this section, we consider the case of a scalar field probing the near-horizon geometry of a Schwarzschild black hole. We shall restrict the analysis to the time-independent modes of the scalar field. The Klein-Gordon operator governing the near-horizon dynamics can then be written as [7]

$$H = -\frac{d^2}{dx^2} + \frac{a}{x^2}, \tag{2.1}$$

where a is a real dimensionless constant, and $x \in [0, \infty]$ is the near-horizon coordinate. For the Schwarzschild background, we have $a = -\frac{1}{4}$. For the moment, however, we can consider a general value of a.

The essential point to note is that the operator H can be factorized as

$$H = A_+ A_-, \tag{2.2}$$

where

$$A_\pm = \pm\frac{d}{dx} + \frac{b}{x}, \tag{2.3}$$

and

$$b = \frac{1}{2} \pm \frac{\sqrt{1 + 4a}}{2}. \tag{2.4}$$

We note that $a = -\frac{1}{4}$ is the minimum value of a for which b is real. For real values of b, A_+ and A_- are formal adjoints of each other (with respect to the measure dx), and consequently H is formally a positive quantity (there are some subtleties to this argument arising from the self-adjoint extensions of H which will be discussed later). When $a < -\frac{1}{4}$, b is no longer real and A_+ and A_- are not even formal adjoints of each other. However, H can still be factorized as in Eqn. (2.2), but it is no longer a positive definite quantity. It can still be made self-adjoint [12], but remains unbounded from below; this case has been analyzed in [13].

Let us now define the operators

$$L_n = -x^{n+1}\frac{d}{dx}, \qquad n \in \mathbf{Z}, \tag{2.5}$$

$$P_m = \frac{1}{x^m}, \qquad m \in \mathbf{Z}. \tag{2.6}$$

In terms of these operators, $A_\pm$ and H can be written as

$$A_\pm = \mp L_{-1} + bP_1, \tag{2.7}$$

$$H = (-L_{-1} + bP_1)(L_{-1} + bP_1). \tag{2.8}$$

Thus, L_{-1} and P_1 are the basic operators appearing in the factorization of H. Taking all possible commutators of these operators between themselves and with H, we obtain the following

relations

$$[P_m, P_n] = 0, \tag{2.9}$$

$$[L_m, P_n] = nP_{n-m}, \tag{2.10}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{2.11}$$

$$[P_m, H] = m(m+1)P_{m+2} + 2mL_{-m-2}, \tag{2.12}$$

$$[L_m, H] = 2b(b-1)P_{2-m} - (m+1)(L_{-1}L_{m-1} + L_{m-1}L_{-1}). \tag{2.13}$$

Eqn. (2.11) describes a Virasoro algebra with central charge c. Note that the algebra of the generators defined in Eqn. (2.5) would lead to $[L_m, L_n] = (m-n)L_{m+n}$. However, this algebra is known to admit a non-trivial central extension. Moreover, for any irreducible unitary highest weight representation of this algebra, $c \neq 0$ [14]. For these reasons, we have included the central term explicitly in Eqn. (2.11).

Eqns. (2.9 - 2.11) describe the semidirect product of the Virasoro algebra with an abelian algebra satisfied by the shift operators $\{P_m\}$. Henceforth, we denote this semidirect product algebra by $\mathcal{M}$. Note that L_{-1} and P_1 are the only generators that appear in H. Starting with these two generators, and using Eqns. (2.12) and (2.13), we see that the only operators which appear are the Virasoro generators with negative index (except L_{-2}), and the shift generators with positive index. Thus, L_m with $m \geq 0$ and P_m with $m \leq 0$ do not appear in the above expressions. In the next section, we will discuss how these quantities are generated.

Although the algebra of Virasoro and shift generators has a semidirect product structure, the operator H however does not belong to this algebra. This is due to the fact that the right-hand side of Eqn. (2.13) contains products of the Virasoro generators. While such products are not elements of the algebra, they do belong to the corresponding enveloping algebra. The given system is thus seen to be described by the enveloping algebra of the Virasoro generators, together with the abelian algebra of the shift operators. This algebraic system has been extensively studied in the literature [10].

3. Representation

We wish to discuss the representation theory of the algebra $\mathcal{M}$, and the implications for the quantum properties of the Klein-Gordon operator H. The eigenvalue equation of interest is

$$H|\psi\rangle = E|\psi\rangle, \tag{3.1}$$

with the boundary condition that $\psi(0) = 0$. We are especially interested in the bound state sector of H. As we have seen, the operator H can be expressed in terms of certain operators that belong to the algebra $\mathcal{M}$. This observation allows us to give a description of the states of H in terms of the representation spaces of $\mathcal{M}$. We first recall the relevant aspects of the representation theory of $\mathcal{M}$.

Following [10], we introduce the space $V_{\alpha,\beta}$ of densities containing elements of the form $P(x)x^\alpha(dx)^\beta$, where α, β are complex numbers, in general. Here, $P(x)$ is an arbitrary polynomial in x and x^{-1}, where x is now treated as a complex variable. It may be noted that the algebra $\mathcal{M}$ remains unchanged even when x is complex. It is known that $V_{\alpha,\beta}$ carries a representation of the algebra $\mathcal{M}$. The space $V_{\alpha,\beta}$ is spanned by a set of vectors, $\omega_m = x^{m+\alpha}(dx)^\beta$,

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where $m \in \mathbf{Z}$. The Virasoro generators and the shift operators have the following action on the basis vectors ω_m,

$$P_n(\omega_m) = \omega_{m-n}, \tag{3.2}$$

$$L_n(\omega_m) = -(m + \alpha + \beta + n\beta)\omega_{n+m}. \tag{3.3}$$

The representation $V_{\alpha,\beta}$ is reducible if $\alpha \in \mathbf{Z}$ and if $\beta = 0$ or 1; otherwise it is irreducible.

The requirement of unitarity of the representation $V_{\alpha,\beta}$ leads to several important consequences. In any unitary representation of $\mathcal{M}$, the Virasoro generators must satisfy the condition $L^\dagger_{-m} = L_m$. In the previous section, we saw that L_{-2} and L_m for $m \geq 0$ did not appear in the algebraic structure generated by the basic operators appearing in the factorization of H. However, the requirement of a unitary representation now leads to the inclusion of L_m for $m > 0$. The remaining generators now appear through appropriate commutators, thus completing the algebra $\mathcal{M}$.

Unitarity also constrains the parameters α and β, which must now satisfy the conditions

$$\beta + \bar{\beta} = 1, \tag{3.4}$$

$$\alpha + \beta = \bar{\alpha} + \bar{\beta}, \tag{3.5}$$

where $\bar{\alpha}$ denotes the complex conjugate of α. Finally, the central charge c in the representation $V_{\alpha,\beta}$ is given by

$$c(\beta) = -12\beta^2 + 12\beta - 2. \tag{3.6}$$

The above representation of $\mathcal{M}$ can now be used to analyze the eigenvalue problem of Eqn. (3.1). We would like to have a series solution to the differential Eqn. (3.1), and consequently choose an ansatz for the wave function $|\psi\rangle$ given by

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n \omega_n. \tag{3.7}$$

Furthermore, the operator H, as written in Eqn. (2.2), has a well-defined action on $|\psi\rangle$. From Eqn. (3.3), it may be seen that

$$L_{-1}(\omega_n) = -(n + \alpha)\omega_{n-1}, \tag{3.8}$$

which is independent of β. Therefore, it appears that an eigenfunction of H may be constructed from elements of $V_{\alpha,\beta}$ for arbitrary β. However, the unitarity conditions of Eqns. (3.4-3.5) put severe restrictions on β, as we shall see below.

The indicial equation obtained by substituting Eqn. (3.7) in Eqn. (3.1) gives

$$\alpha = b, \text{ or } (1 - b). \tag{3.9}$$

To proceed, we analyze the cases (i) $a \geq -\frac{1}{4}$, and (ii) $a < -\frac{1}{4}$ separately.

(i) $a \geq -\frac{1}{4}$.

This is the main case of interest as it includes the value of a for the Schwarzschild background. It follows from Eqn. (2.4) and Eqn. (3.9) that b and α are real. The unitarity condition of Eqn (3.4) now fixes the value of $\beta = \frac{1}{2}$, and the corresponding central charge is given by

$c = 1$. It may be noted that relation of the central charge calculated here to that appearing in the calculation of black hole entropy depends on geometric properties of the black hole in question. We do not address this issue here. Thus, we see that for the Schwarzschild black hole, we have identified the relevant representation space as $V_{1/2,1/2}$.

(ii) $a < -\frac{1}{4}$.

In this case, we can write $a = -\frac{1}{4} - \mu^2$ where $\mu \in \mathbf{R}$. It follows from Eqn. (2.4), that $b = \frac{1}{2} \pm i\mu$. Eqn. (3.9) then gives $\alpha = \frac{1}{2} \pm i\mu$, or $-\frac{1}{2} \mp i\mu$. Let us take the case when $\alpha = \frac{1}{2} + i\mu$, the other cases being similar. From Eqns. (3.4) and (3.5), we find $\beta = \frac{1}{2} - i\mu$. The value of the corresponding central charge is given by $c = 1 + 12\mu^2$. The operator H in this case can be made self-adjoint but its spectrum remains unbounded from below [12, 13]. The algebraic description, however, always leads to a well-defined representation.

We return now to the eigenvalue problem for the differential operator H, and focus attention on the Schwarzschild background, for which $a = -\frac{1}{4}$. As already mentioned, we are interested in the bound state sector of H. These states have negative energy and satisfy Eqn. (3.1) with energy $-E$, where $E > 0$. The wave function satisfies the boundary condition $\psi(0) = 0$. This may seem contradictory to the statement in section 2, which claimed that H as written in Eqn. (2.2) is a positive quantity. The resolution of this apparent paradox is as follows. It is known that the operator H admits a one-parameter family of self-adjoint extensions labelled by a $U(1)$ parameter e^{iz}, where z is real [15, 16, 7]. For $a = -\frac{1}{4}$, there is an infinite number of bound states for a given self-adjoint extension z. In all these cases, A_+ and A_- are not adjoints of each other, and consequently H is not a positive quantity. The eigenfunctions and eigenvalues of H in this case are given by [7]

$$\psi_n(x) = N_n \sqrt{x} K_0 \left(\sqrt{E_n} x \right), \tag{3.10}$$

$$E_n = \exp \left[\frac{\pi}{2} (1 - 8n) \cot \frac{z}{2} \right], \tag{3.11}$$

where n is an integer, N_n is a normalization factor, and K_0 is the modified Bessel function.

We have thus shown how to obtain the spectrum of H using the representation of the algebra $\mathcal{M}$. In the next section, we shall analyze the properties of this spectrum in the near-horizon region.

4. Scaling Properties

As we have seen, the Virasoro algebra plays an important role in determining the spectrum of H. Since this operator is associated with a probe of the near-horizon geometry, one might expect that the corresponding wave functions would exhibit certain scaling behaviour in this region.

Firstly, let us recall that the horizon in this picture is located at $x = 0$. However, the wave functions ψ_n vanish at $x = 0$, and therefore do not exhibit any non-trivial scaling. Nevertheless, it is of interest to examine the behaviour of the wave functions near the horizon. For $x \sim 0$, the wave functions have the form

$$\psi_n = N_n \sqrt{x} \left(A - \ln \left(\sqrt{E_n} x \right) \right), \tag{4.1}$$

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where $A = \ln 2 - \gamma$, and γ is Euler's constant [17]. While the logarithmic term, in general, breaks the scaling property, one notices that it vanishes at the point $x_0 \sim 1/\sqrt{E_n}$, where the wave functions exhibit a scaling behaviour. The entire analysis so far, including the existence of the Virasoro algebra, is valid only in the near-horizon region of the black hole. Therefore, consistency of the above scaling behaviour requires that x_0 belongs to the near-horizon region. The minimum value of x_0 is obtained when E_n is maximum. When the parameter z appearing in the self-adjoint extension of H is positive, the maximum value of E_n is given by

$$E_0 = \exp\left[\frac{\pi}{2}\cot\frac{z}{2}\right] . \tag{4.2}$$

However, when z is negative, the maximum value of E_n is obtained when $n \to \infty$. In this case, $x_0 \to 0$ where, as we have seen before, the wave function vanishes and scaling becomes trivial. We therefore conclude that

$$x_0 \sim \frac{1}{\sqrt{E_0}}, \quad z > 0 \tag{4.3}$$

is the minimum value of x_0. It remains to show that x_0 given by Eqn. (4.3) belongs to the near-horizon region. We first note that we are free to set z to an arbitrary positive value. Thus, we consider $z > 0$ such that $\cot\frac{z}{2} >> 1$; this is achieved by choosing $z \sim 0$. For all such z, we find that x_0 is small but nonzero, and thus belongs to the near-horizon region. In effect, we can use the freedom in the choice of z to restrict x_0 to the near-horizon region.

We now consider a band-like region $\Delta = [x_0 - \delta/\sqrt{E_0}, x_0 + \delta/\sqrt{E_0}]$, where $\delta \sim 0$ is real and positive. The region Δ thus belongs to the near-horizon region of the black hole. At a point x in the region Δ, the leading behaviour of ψ_n is given by

$$\psi_n = N_n\sqrt{x}\left(A + 2\pi n\cot\frac{z}{2}\right) . \tag{4.4}$$

Thus, all the eigenfunctions of H exhibit a scaling behaviour, i.e. $\psi_n \sim \sqrt{x}$, in the near-horizon region Δ. It should be stressed that this analysis is made possible by utilizing the freedom in the choice of z. The parameter z, which labels the self-adjoint extensions of H, thus plays a crucial role in establishing the self-consistency of this analysis.

We conclude this section with the following remarks:

1. A particular choice of z is equivalent to a choice of domain for the differential operator H. Physically, the domain of an operator is specified by boundary conditions. A specific value of z is thus directly related to a specific choice of boundary conditions for H. Thus, we see that the system exhibits non-trivial scaling behaviour only for a certain class of boundary conditions. These boundary conditions play a conceptually similar role to the fall-off conditions as discussed in Ref. [2, 3].

2. The analysis above provides a qualitative argument which suggests that the scaling behaviour in the presence of a black hole should be observed within a region Δ. Although Δ belongs to the near-horizon region of the black hole, it does not actually contain the event horizon. Our picture is thus similar in spirit to the stretched horizon scenario of Ref. [11].

5. Conclusion

In this paper, we have analyzed the near-horizon properties of the Schwarzschild black hole, using a scalar field as a simple probe of the system. We restricted attention to the time-independent modes of the scalar field, and this allowed us to obtain a number of interesting results regarding the near-horizon properties of the black hole. It is possible that more sophisticated probes of general field configurations may lead to additional information.

The factorization of H, leading to the algebraic formulation of section 2, is a process which appears to be essentially classical. However, the central charge in the algebra $\mathcal{M}$ goes beyond the classical framework, as it arises from the requirement of a non-trivial representation. As discussed, the algebra appearing in Eqns. (2.9-2.13) does not at first contain all the Virasoro generators. The requirement of unitarity of the representation leads to the inclusion of all the generators. It is thus fair to say that the full Virasoro algebra appears in our framework only at the quantum level. The operator H does not belong to $\mathcal{M}$ but is contained in the enveloping algebra of the Virasoro generators. The enveloping algebra is the natural tool that is used to obtain representations of $\mathcal{M}$. Thus, even though H is not an element of $\mathcal{M}$, it nevertheless has a well-defined action in any representation of $\mathcal{M}$. It is this feature that makes the algebraic description useful.

In section 3, we summarized some results from the representation theory of $\mathcal{M}$. The operator H is now treated at the quantum level, and the corresponding eigenvalue problem is studied using the representations of $\mathcal{M}$. Unitarity again plays a role in restricting the space of allowed representations. It is interesting to note that for all values of the coupling $a \geq -\frac{1}{4}$, the value of the central charge in the representation of $\mathcal{M}$ is equal to 1. Other black holes which have a in this range would exhibit a universality in this regard. As mentioned in section 3, the relationship of c calculated here to that appearing in the entropy calculation of a particular black hole would depend on other factors which are likely to break the universality.

If a Virasoro algebra is associated with the near-horizon dynamics, then some reflection of it should appear in the spectrum of H. In particular, we can expect that the wave functions of H in the near-horizon region should exhibit scaling behaviour. Such a property was indeed found in a band-like region near the horizon. It is interesting to note that this band excludes the actual horizon. This is similar in spirit to the stretched horizon scenario of black hole dynamics. The parameter z describing the self-adjoint extensions of H is restricted to a set of values in this process. This implies that the near-horizon wave functions exhibit scaling behaviour only for a certain class of boundary conditions. It is important to note that boundary conditions also played a crucial role in proving the existence of a Virasoro algebra in Ref. [2, 3]. This feature provides a common thread in these different approaches towards the problem.

It is known that the near-horizon dynamics of various black holes is described by an operator of the form H [7, 5], for different values of $a \geq -\frac{1}{4}$. Any such operator can be factorised as in Eqn. (2.2) and the above analysis will also apply to these black holes. It has been claimed in [2, 3] that a Virasoro algebra is associated with a large class of black holes in arbitrary dimensions. It seems plausible that the near-horizon dynamics of probes in the background of these black holes would be described by an operator of the form of H.

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